LOCAL THICKNESS OF A WASHING FILM IN THE ENTRANCE SECTION

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A method of graphic analysis is considered for the calculation of the local thickness of a liquid film in the entrance section of the transformation of slot flow into film flow over a vertical plate. The proposed method can also be used in calculating the local thicknesses of a thin layer during liquid flow over curved surfaces.

The wide application of film flow in modern technology [1-4] poses a number of problems involving the properties of the hydrodynamics of flow in a film. In the washing of vertical and inclined walls by a thin layer of liquid a smooth entrance section of transformation of slot flow into film flow with a length x_0 (Fig. 1) and a section of steady wave flow of the film are clearly distinguished. In Fig. 1, 1 is the washed surface, 2 is the slot, and 3 is the surface of the film. One important problem is the determination of the local thickness of the washing film in the entrance section.

The solution of this problem is necessary for the establishment of the region of existence of film flow (an anayltical calculation of the minimum washing density for the given conditions [1]) and for the calculation of the local coefficients of heat and mass transfer [1, 2, 4] in the entrance section. The experimental and theoretical solution of this problem is difficult since the thickness δ of the washing liquid layer depends on the flow rate of the liquid, the form and curvature of the washed surface, and a number of other factors. It has not been possible to obtain a general solution allowing for all of these factors [1, 4], although such a solution can be obtained for individual cases (the washing of a vertical plate, vertical tubes, etc.).

As is known [5], the geometrical characteristics of the flow figure in the equations of continuity of the medium, which can be written in the general form

$$\int_F w_x dF = \overline{w}F = \text{const} \tag{1}$$

where w_x is the local velocity in the direction of flow, F is the area of a cross section of the stream, and w is the mean flow velocity.

To convert Eq. (1) to a more convenient form let us use the dimensionless values

$$u = \overline{w}_x / w_S, \; y = (\overline{w}_S / \nu S)^{0.5} y_i, \; x = x_i / S \tag{2}$$

where it is assumed that w_S is the mean velocity of the liquid at the exit from the slot of the distributing device, S is the width of the slot, and x_i, y_i are the coordinates of the plane in which the flow takes place (Fig. 1).

In the further transformation of (1) we will assume that the film flow in the entrance section is laminar and waves are absent at the free surface of thefilm.

Inserting the values of (2) into Eq. (1) we find that for a flat washed surface the equation of continuity of the medium takes the form

$$\int_0^{\delta^*} u dy = 1S^* = 0.5\,\mathrm{Re}^{0.5} \tag{3}$$

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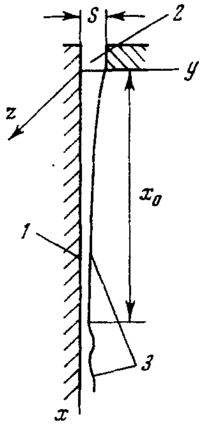

Fig. 1

where $S^* = S(w_S/\nu S)^{0.5}$ is the dimensionless width of the distributnig slot; $Re = 4w\delta/\nu$ is the Reynolds number for film flow.

The solution of Eq. (3) relative to δ is complicated by the fact that one cannot always determine the integral on the left side. The determination of $\int_0^{\delta^*} u dy$ is connected with finding the law of velocity distribution over the cross section of the film in the entrance section of its flow. For the stabilized flow, when $u = u(y)$ this problem is simpler than for the case of the entrance section, where $u = u(x, y)$. The determination of δ for the entrance section is of great interest since the conditions affecting the nature of the film flow downstream arise in the entrance section.

An equation of motion having the following from [1] is valid in general for flow in the entrance section:

$$dw_x / d\tau = \nu (\partial^2 w_x / \partial x^2 + \partial^2 w_x / \partial y^2) + f_v \tag{4}$$

where τ is the time and f_V is a volumetric force per unit of density and directed along the x axis (acceleration of free fall).

The solution of Eq. (4) in general form by analytical means presents great difficulties. As shown in [6], as a first approximation with a degree of convergence of the results which is satisfactory in practice the flow in the initial section of the film can be described by an equation which with allowance for (2) has the following dimensionless form:

$$\frac{\partial^2 u}{\partial y^2} - u\frac{\partial u}{\partial x} = -f_v^*, \quad f_v^* = \frac{f_v S}{\bar{w}_S^2} \tag{5}$$

This nonlinear differential equation can be solved in the following way.

We assume thta the function u(x, y) has the form

$$u = \varphi_0(y) + (x+\varepsilon)^{-1}\varphi_1(y) + (x+\varepsilon)^{-2}\varphi_2(\varphi) + (x+\varepsilon)^{-3}\varphi_3(y) + \dots \tag{6}$$

where ε = const and φ_i are functions as yet unknown.

Substituting Eq. (6) into (5) and collecting the terms having common multiples of $(x + \varepsilon)^k$ one can obtain a system of differential equations relative to the functions $\varphi_0(y)$, $\varphi_1(y)$, $\varphi_2(y)$, $\varphi_3(y)$,... Integration of these equations yields the following equations for φ_i:

$$\varphi_0(y) = -f_v^* \frac{y^2}{2} + C_0 y, \quad \varphi_1(y) = C_1 y \tag{7}$$

$$\varphi_2(y) = \frac{C_1}{40} f_v^* y^5 - \frac{C_0 C_1}{12} y^4 + C_2 y$$

$$\varphi_3(y) = f_v^* C_2 \frac{y^5}{20} - (C_1^2 + 2C_0C_2)\frac{y^4}{12} + C_3 y$$

$$\varphi_4(y) = -\frac{C_0 C_3 y^4}{4} + \frac{f_v^* 3C_3}{40} y^5 - \frac{C_1 C_2}{4} y^4$$

$$\varphi_5(y) = -\frac{(C_2^2 + 2C_1C_3)}{6} y^4, \quad \varphi_6(y) = -\frac{5}{12} C_2 C_3 y^4$$

As an analysis showed, beginning with $\varphi_7(y)$ etc. the functions $\varphi_i(y)$ are small values and can be neglected in the case under consideration.

The integration constants entering into (7) are determined from the boundary conditions at the surface of the film and outside the limits of the initial section. For example, from the condition at $x = \infty$ one can find that

$$C_0 = [6 Re^{0.5} Fr_S^{-2}]^{1/3}, \quad Fr_S = \bar{w}_s / (f_v S)^{0.5}$$

The values C_1, C_2, and C_3 have a more complicated form and are determined from the condition of least deviation from the real solution of Gauss [7], according to which the velocity increment $p = u(x, y)|_{x=0} - u$ in the initial section can be determined from the condition

$$\int_0^{0.5 Re^{0.5}} p^2 dy = \min \tag{8}$$

Satisfying the requirement of a minimum gives

$$\frac{\partial}{\partial C_i} \int_0^{0.5 Re^{0.5}} p^2 dy = \int_0^{0.5 Re^{0.5}} p \frac{\partial p}{\partial C_i} dy = 0 \tag{9}$$

In accordance with (6) and the expression for u(x, y) at the exit from the distributing slot (1) one can write for p

$$p = y\left[C_0 + \frac{C_1}{\varepsilon} + \frac{C_2}{\varepsilon^2} + \frac{C_3}{\varepsilon^3} - 6(\mathrm{Re})^{-0.5}\right] + y^2\left[\frac{6}{\mathrm{Re}} - \frac{1}{2\,\mathrm{Fr}}\right] - $$
$$- y^4\left[\frac{C_0C_1}{12\varepsilon^2} + \frac{(C_1^2+2C_0C_2)}{12\varepsilon^3} + \frac{1}{4\varepsilon^4}(C_1C_2+C_0C_3) + \frac{1}{6\varepsilon^5}(C_2^2+2C_1C_3) + \frac{5C_3C_2}{12\varepsilon^6}\right] + y^5\left[\frac{f_v{}^*C_1}{40\varepsilon^2} + \frac{f_v{}^*C_2}{20\varepsilon^3} + \frac{f_v{}^*3C_3}{40\varepsilon^4}\right] \tag{10}$$

Substituting (10) into (9) we obtain a system of four nonlinear equations. If the new variables

$$a_1 = C_0 + \frac{C_1}{\varepsilon} + \frac{C_2}{\varepsilon^2} + \frac{C_3}{\varepsilon^3} - 6(\mathrm{Re})^{-0.5}$$
$$a_4 = -\left[\frac{C_0C_1}{12\varepsilon^2} + \frac{(C_1^2+2C_0C_2)}{12\varepsilon^3} + (4\varepsilon^4)^{-1}(C_1C_2+C_0C_3) + (6\varepsilon^5)^{-1}(C_2^2+2C_1C_3) + \frac{5}{12}\frac{C_2C_3}{\varepsilon^6}\right] \tag{11}$$
$$a_5 = \frac{f_v{}^*C_1}{40\varepsilon^2} + \frac{f_v{}^*C_2}{20\varepsilon^3} + \frac{3f_v{}^*C_3}{40\varepsilon^4}, \quad K = \left(\frac{6}{\mathrm{Re}} - \frac{1}{2\,\mathrm{Fr}}\right)$$

are used one can obtain a simpler system of equations.

With the notation (11) the value p will equal

$$p = a_1y + Ky^2 + a_4y^4 + a_5y^5 \tag{12}$$

The transformation (11) is possible in the case where the functional determinant of the transformation takes on values not equal to zero. In this case we have

$$F = \begin{vmatrix} \frac{\partial a_1}{\partial C_1} & \frac{\partial a_1}{\partial C_2} & \frac{\partial a_1}{\partial C_3} \\ \frac{\partial a_4}{\partial C_1} & \frac{\partial a_4}{\partial C_2} & \frac{\partial a_4}{\partial C_3} \\ \frac{\partial a_5}{\partial C_1} & \frac{\partial a_5}{\partial C_2} & \frac{\partial a_5}{\partial C_3} \end{vmatrix} = \frac{f_v{}^*6C_3}{480\varepsilon^6} \neq 0 \tag{13}$$

Consequently, the transformation adopted is admissible.

With allowance for the notation (11) the Gauss requirement is written in the form of three algebraic equations relative to a_i. The unknown a_1, a_4, and a_5 are determined from these equations by the Cramer rule. Then the transition is made to the values C_1, C_2, and C_3.

Substituting Eq. (7) into Eq. (6), we obtain the law of velocity distribution in the entrance section of film flow in the form

$$u = \left[C_0y - f_v{}^*\frac{y^2}{2}\right] + (x+\varepsilon)^{-1}C_1y + (x+\varepsilon)^{-2}\left[\frac{f_v{}^*C_1}{40}y^5 - \frac{C_0C_1}{12}y^4 + C_2y\right] + (x+\varepsilon)^{-3}\left[\frac{f_v{}^*C_2}{20}y^5 - (C_1^2+2C_0C_2)\frac{y^4}{12} + \right.$$
$$\left. + C_3y\right] + (x+\varepsilon)^{-4}\left[\frac{3f_v{}^*C_3}{40}y^5 - \frac{C_0C_3+C_1C_2}{4}y^4\right] + (x+\varepsilon)^{-5}\left[-\frac{C_2^2+2C_1C_3}{6}y^4\right] + (x+\varepsilon)^{-6}\left[-\frac{C_2C_35}{12}y^4\right] \tag{14}$$

The constant ε entering into Eq. (14) is determined on the basis of Eq. (3) for x = 0, i.e, through the given flow rate of liquid in the distributing device.

Knowing the law of velocity distribution over the cross section of the flowing film one can integrate the equation of continuity of the medium (3) in the limits of variation of y from 0 to δ^* and obtain the following equation:

$$\frac{C_0}{2}(\delta^*)^2 - \frac{f_v{}^*}{6}(\delta^*)^3 + \frac{C_1}{2(x+\varepsilon)}(\delta^*)^2 + (x+\varepsilon)^{-2}\left[\frac{f_v{}^*C_1}{240}(\delta^*)^6 - \frac{C_0C_1}{60}(\delta^*)^5 + \frac{C_2}{2}(\delta^*)^2\right] + (x+\varepsilon)^{-3} \times$$
$$\times \frac{f_v{}^*C_2}{120}(\delta^*)^6 - \frac{C_1^2+2C_0C_2}{60}(\delta^*)^5 + 0.5C_3(\delta^*)^2\Big] + (x+\varepsilon)^{-4}\left[\frac{3f_v{}^*C_3}{240}(\delta^*)^6 - \frac{C_0C_3-C_1C_2}{20}(\delta^*)^5\right] + \tag{15}$$
$$+ (x+\varepsilon)^{-5}\frac{2C_1C_3+C_2^2}{30}(\delta^*)^5 - (x+\varepsilon)^{-6}\frac{C_2C_3}{12}(\delta^*)^5 = 0.5\,\mathrm{Re}^{0.5}$$

The algebraic equation obtained must be solved relative to the value $\delta^* = (w_S/\nu S)^{0.5}\delta$. In connection with the fact that Eq. (15) is complicated and its analytical solution has a cumbersome form, it is convenient to obtain the value of δ^* by using a graphic method.

In the graphic method of solving Eq. (15) for a specific value of x to which the film thickness is sought one must construct in the system of coordinates $\int_0^{\delta^*} u\,dy - y$ a curve characterizing the left side of Eq. (15) by substituting the values of y in place of δ^*. Since at a certain value y = δ^* the left side of Eq. (15) is equal to the right side, a point is sought on the ordinate axis whose value corresponds to $0.5\mathrm{Re}^{0.5}$.

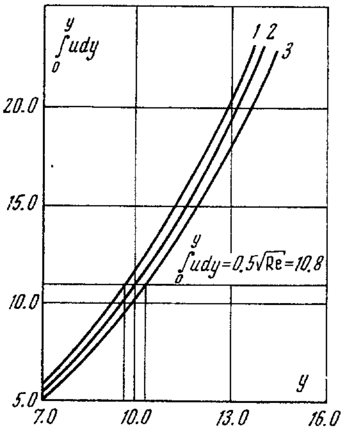

Fig. 2

The point of intersection of the curve $\int_0^{\delta^*} udy - y$ with a line parallel to the abscissa and passing at a distance of $0.5Re^{0.5}$ from this axis gives the value of δ^* sought.

As an example, let us determine the thickness of a film formed by a water stream with a temperature t = 80° C which flows from a flat-slotted distributing device with a width $S = 2 \cdot 10^{-3}$ m. It is assumed that the Reynolds number in this case is Re = 465. This means that the mean velocity of discharge from the slot is $w_S = 0.25$ m/sec, while $S^* = 0.5Re^{0.5} \approx 10.8$.

For the given case the constants C_0, C_1, C_2, and C_3 entering into Eq. (15) are

$$C_0 = 14.8847, \quad C_1 = -0.119296$$
$$C_2 \approx +0.0777034, \quad C_3 \approx +0.0612 \tag{16}$$

The functions φ_0, φ_1, φ_2, φ_3, φ_4, and φ_5 and φ_6 can be represented in the form

$$\begin{aligned} &\varphi_0(y) = 0.5f_v{}^*y^2 + 14.8847y \\ &\varphi_1(y) \approx -0.119296y \\ &\varphi_2(y) \approx -0.0002f_v{}^*y^5 + 0.147932\,y^4 + 0.0777034y \\ &\varphi_3(y) = 0.00038f_v{}^*y^5 - 0.1226y^4 + 0.0612y \\ &\varphi_4(y) = -0.227507\,y^4 + 0.0046\,y^5, \quad \varphi_5(y) = 0.00014\,y^4 \\ &\varphi_6(y) = 0.00001\,y^4 \end{aligned} \tag{17}$$

On the basis of the values of the functions $\varphi_i(y)$ (i = 0, 1, 2, 3, 4, 5, 6) presented in Eqs. (17) the law of velocity distribution (6) takes the form

$$\begin{aligned} u(x, y) = -0.5\,f_v{}^*y^2 + 14.8847\,y - 0.119296\,(x+\varepsilon)^{-1}y + \\ + (x+\varepsilon)^{-2}(0.0777034\,y + 0.147932\,y^4 - 0.0002\,f_v{}^*y^5) + (x+\varepsilon)^{-3}(0.00038\,f_v{}^*y^5 - 0.1226y^4 + 0.0612y) \\ + (x+\varepsilon)^{-4}(0.0046y^5 - 0.2275071\,y^4) + (x+\varepsilon)^{-5}\,0.00014\,y^4 + (x+\varepsilon)^{-6}\,0.00001\,y^4 \end{aligned} \tag{18}$$

Having integrated Eq. (18) with respect to y in the limits from 0 to δ^* and having taken specific values of x (for example, x_1, x_2, x_3, x_4), one can construct a system of curves describing the function

$$\int_0^{\delta^*} u(x, y)\,dy.$$

Such curves are presented in Fig. 2 for values of x of 1.0, 0.5, and 0.25 (curves 1, 2, and 3, respectively). Since according to the equation of continuity the curves constructed determine a function which at certain values of δ^* (unknown in the present case) must equal $0.5Re^{0.5}$, a point with the value $0.5Re^{0.5} = 10.8$ is found on the same ordinate axis and a curve parallel to the abscissa is drawn through it. The point of intersection of this line with the curves constructed is a unique point satisfying the equation of continuity of the medium (15) and it determines the film thickness at the given value of x_i. For instance, in the example considered we find that $\delta^* = 8$ at x = 1, $\delta^* = 10$ at x = 0.5, etc. This means that at the given values of x the following results are obtained:

$$\begin{aligned} &x = 1, \quad (\overline{w}_S / \nu S)^{0.5}\,\delta = 8 \\ &x = 0.5, \quad (\overline{w}_S / \nu S)^{0.5}\,\delta = 10 \end{aligned} \tag{19}$$

By solving equations of the type (19) we can determine the concrete values (in mm) of the film thickness δ. From these values one can trace the variation in δ along the flowing liquid film, i.e, find $\delta = \delta(x)$.

This method can also be used in a case when one is considering flow not on a flat but on a curved surface, for example cylindrical. In this case one must take into account those changes which are inherent to Eqs. (3) and (14) and to Reynolds number for film flow over a curved surface [1].

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